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MACHINE CONTOURING USING MINIMUM CURVATURE

by

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ABSTRACT

Machine contouring must not introduce information which is not present in the data. The one-dimensional spline fit has well defined smoothness properties. These are duplicated for two-dimensional interpolation in this paper. The problems of a piecewise polynomial fit for randomly spaced data are avoided by solving the corresponding differential equation. Finite difference equations are deduced from a principle of minimum total curvature, and an iterative method of solution is outlined. Gravity and aeromagnetic surveys provide examples which compare favourably with the work of draughtsmen.

INTRODUCTION

Contour maps are useful in the evaluation and interpretation of geophysical data. With the rapid increase in the rate of acquisition of data, a computer becomes an attractive means of producing contour maps.

Although errors occur in most geophysical observations, contour maps are usually drawn so that the imaginary surface on which the contours lie passes exactly through the observations. The problem of interpolation is then either: (a) to define a continuous function of the two space variables, which takes the values of the observations at the required, perhaps random, positions; or (b) to define a set of values at the points of a regular grid, so that a grid point value tends to an observational value if the position of the observation tends to the grid point. A solution to (a) gives a solution to (b), but a solution to (b) may not give a solution to (a). The solution to (b) is most commonly the one used as an input to a program which draws contour lines.

Methods for the production of contour maps have been published by Crain & Bhattacharyya (1967), Smith (1968), Cole (1968), Pelto, Elkins & Boyd (1968), and McIntyre, Pollard & Smith (1968). These methods are variations of either weighting or function fitting or both, and give a solution to problem (a) and hence (b). Crain (1970) has provided a review of these methods.

This article describes a method for finding a solution to problem (b) without first finding a solution to problem (a). The solution also happens to be the smoothest. This attribute gives confidence in the use of the method and explains the quality of the resulting contour maps.

The problem of interpolation in one dimension has led to the piecewise polynomial fit, or spline (Ahlberg, Nilson & Walsh, 1967). A continuous function is found for all values of the independent variable. This method has been extended to two dimensions (De Boor, 1962), and used by Bhattachryya (1969) to give a solution to problem (a).

However, if the observation points in two dimensions are randomly situated, the fitting of piecewise two-dimensional polynomials to polygons seems difficult, although it is possible if the set of polygons are topologically equivalent to a rectangular grid (Hessing, Lee, Pierce & Powers, 1972).

The optimum properties of the spline fit can be obtained in both one and two dimensions by solving the differential equation equivalent to a third-order spline. This is the equation which describes the displacement of a thin sheet in one or two dimensions under the influence of point forces. The 'boundary conditions' are not only at the ends or boundary, but within the region of interest. The solution is forced to take up the value of the observation at the point of observation, in one or two dimensions. The equation is solved numerically, and thus gives a solution to problem (b).

The smoothness properties follow from the method of deducing the difference equations, and the quality of the resulting contour map is thus determined. The solution of the set of difference equations is a time-consuming process, but the iteration times on the computer have been reduced and can be reduced still further.

THE METHOD

The differential equation

The thin metallic strip or sheet is bent by forces acting at points so that the displacement at these points is equal to the observation to be fitted. Let u be the displacement, x, y the space variables, and let forces f_n act at (x_n , y_n), n = 1,..,N, where the observations are w_n, then (Love, 1926)

$$\frac{d^4u}{dx^4} = f_n \, , \quad x = x_n \, ,$$

$$= 0 \text{ otherwise,} \qquad (1)$$

in one dimension and

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f_n \, , \quad x = x_n \, , \; y = y_n \, ,$$

$$= 0 \text{ otherwise,} \qquad (2)$$

in two dimensions. The units are dimensionless. A condition on the solution is that $u(x_n) = w_n$ or $u(x_n, y_n) = w_n$. In one dimension, u, $\frac{\partial u}{\partial x}$, and $\frac{\partial^2 u}{\partial x^2}$, the curvature, are continuous across the point where the force is acting, but $\frac{\partial^3 u}{\partial x^3}$ is discontinuous across such a point and the value of the discontinuity is equal to the force acting at that point (Love, 1926). A solution in one dimension is given by a third-order polynomial

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \, ,$$

for each segment between the points where the forces are acting. The coefficients $a_0, .., a_3$ are found by using the continuity conditions above. This solution is a cubic spline.

In two dimensions the solution of equation (2) is to be used in place of the two-dimensional third-order piecewise polynomial fit.

Boundary conditions

The most suitable condition for the ends of the strip or edge of the thin sheet is that of freedom. For a strip, the region between the end and the extreme observation will have a linear form, and for a sheet, the area between the edge and the observations will tend to a plane, as the sheet becomes larger.

For one and two dimensions, at the ends or edge, the force is zero, and the bending moment about a tangential line is zero. For one dimension, these conditions give

$$\frac{\partial^3 u}{\partial x^3} = 0 \qquad (3)$$

and

$$\frac{\partial^2 u}{\partial x^2} = 0 \text{ , respectively} \qquad (4)$$

For two dimensions, they give

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \quad , \qquad (5)$$

where the normal to the edge is in the x direction, and give (4) also. The condition that

$$u(x_n, y_n) = w_n \qquad (6)$$

is also a 'boundary' condition.

Equation (1) with boundary conditions (3), (4), and (6) or equation (2) with boundary conditions (5), (4), and (6) are solved numerically.

Finite difference equations

There are two reasons for not using Taylor's theorem (Young, 1962), to write approximations to (1) and (2). The first is that the finite difference equations equivalent to the boundary conditions (4) and (5) are difficult to determine, and the second is that the proof that the solution defines the smoothest possible surface is simplified if another starting point is used.

The better start is to construct the equivalent of the total squared curvature,

$$C = \iint \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^2 dx\, dy ,$$

directly in terms of elements of the set of grid point values

$$u_{i,j} \equiv u(x_i, y_j) ,$$

$$x_i = ih , \quad y_j = jh , \quad i = 1, \ldots, I , \quad j = 1, \ldots, J ,$$

where h is the grid spacing. A continuous function u(x,y) does not have to exist. The discrete total squared curvature is

$$C = \sum_{i=1}^{I} \sum_{j=1}^{J} \left(c_{i,j} \right)^2 \tag{7}$$

where C_{ij} is the curvature at (x_i, y_j). C_{ij} is a function of $u_{i,j}$ and some neighbouring grid values; the exact set depends on the accuracy with which the curvature is to be represented.

To minimize the sum C, the functions

$$\frac{\partial C}{\partial u_{i,j}}, \quad i = 1, \ldots, I; \; j = 1, \ldots, J, \tag{8}$$

are set equal to zero (Stiefel, 1963). The resulting equations determine a set of relations between neighbouring grid point values, one relation for each grid point.

In one dimension the simplest approximation to the curvature at x_i is

$$\left(u_{i+1} + u_{i-1} - 2 u_i \right) / h^2 ,$$

and in two dimensions at (x_i, y_i), it is

$$C_{i,j} = \left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4\,u_{i,j} \right) / h^2 . \tag{9}$$

Along edges and rows one from the edge, and near corners, different expressions for the curvature are used. For example, at an edge j = 1,

$$C_{i,j} = \left(u_{i+1,j} + u_{i-1,j} - 2\,u_{i,j} \right) / h^2 . \tag{10}$$

These special cases are also included in the total for C. Away from the edges, (9) shows that a grid point value $u_{i,j}$ occurs in the expressions for,

$$C_{i,j} \;,\; C_{i+1,j} \;,\; C_{i-1,j} \;,\; C_{i,j+1} \;, \text{ and } C_{i,j-1} .$$

Thus, only these need be considered when equation (8) is used.
Using (7), (8), and (9) the common difference equation for the biharmonic equation results:

$$\begin{aligned} & u_{i+2,j} + u_{i,j+2} + u_{i-2,j} + u_{i,j-2} \\ & + 2\left(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} \right) \\ & - 8\left(u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1} \right) \\ & + 20\, u_{i,j} = 0 . \end{aligned} \tag{11}$$

For the edge j = 1, the difference equation is

$$u_{i-1,j} + u_{i+2,j} + u_{i,j+2} + u_{i-1,j+1} + u_{i+1,j+1}$$
$$- 4 (u_{i-1,j} + u_{i,j+1} + u_{i+1,j})$$
$$+ 7 u_{i,j} = 0 , \qquad (12)$$

and four remaining types of equations are similarly derived.

The point boundary conditions (6) are used by setting $u_{i,j} = w_n$ wherever $u_{i,j}$ occurs in the set of linear equations, and by removing those equations which correspond to these fixed grid points.

Observation not on a grid point

If an observation does not fall on a grid point, modifications must be made to those equations corresponding to the grid points nearby. These modifications are complicated; the essential method and results are not changed by ignoring them, and so in this article all observations will be assumed to be located on grid points, except in the examples of real data.

Iteration matrix

The set of linear algebraic equations (11), (12), and others are best solved iteratively (Young, 1962). Given an approximate set of $u_{i,j}$, a new set is obtained by making $u_{i,j}$ the subject of equations (11) and (12) and others.

For example, (13) gives

$$u^{p+1}_{i,j} = \left[4 (u^p_{i-1,j} + u^p_{i,j+1} + u^p_{i+1,j}) - (u^p_{i-1,j} + u^p_{i+2,j} + u^p_{i,j+2} + u^p_{i-1,j+1} + u^p_{i+1,j+1}) \right] / 7 , \qquad (13)$$

where the index p indicates the pth iteration. Starting values must be given, and one suitable method is to use the value of the nearest observation or a weighted sum of neighbouring observations.

Iteration matrices which give faster rates of convergence than that defined by (13) are known (Young, 1962; Parter, 1959), but are not described here. The proof of the existence of a solution to the linear equations is omitted (Stiefel, 1963).

Smoothness properties

The measure of smoothness, $C = (C_{i,j})^2$ is a function of h and the precision of the approximation for $C_{i,j}$. Because the linear equations are deduced from the principle of minimum C, for a given h and for a given definition of curvature, the resulting grid point surface is smoother than, or as smooth as, any other grid point surface.

Nothing will be said here about the convergence of the grid point values or of C, as the grid spacing tends to zero. However, for a given grid spacing, the method gives the smoothest possible contour map, and it can be used with some confidence as a representation of the given data.

Drawing lines

There are many different methods of drawing the contours once the grid surface had been found (Crain, 1970). The method used in the examples involves a four-point cubic interpolation between grid points to find contour cuts, and then a cubic spline to join the cuts. The observations are not used. This nonlinear method gives better results than linear interpolation for the same grid spacing.

EXAMPLES

For each map the time for one iteration for one grid point was approximately 0.4 ms using a CDC 3600 computer. Up to two hundred and sixty thousand grid points have been used to contour sixty thousand observations at one time. To provide edge matching when an entire survey cannot be contoured at once, data beyond the area to be contoured are used.

Two simple test examples are; (1) a one-dimensional set of data taken to lie on a straight line; and (2) a set of data points (at least four are necessary) taken to lie on a plane. In case (1) the free grid points tend to values lying on the same straight line, and in case (2) the free grid points tend to values lying in the same plane. These and other higher-order surfaces test the method in general and the difference equations in particular. The illustrated examples use real data.

The iterations were discontinued when all significant relocation of contour lines had taken place.

Almost uniform data

Figure 1 is the resulting contour map for gravimetric data sampled in mgal on a nominal 11 km network. The grid spacing was 1.85 km and the number of iterations was 90. The number of grid points was 1500. The smoothness of the interpolating grid surface is apparent. The shapes of contours for data of this type generally agree with those of draughtsmen. Differences occur when the interpolating grid surface lies outside the range of a closed group of observations.

A deficiency in the line-drawing routine shows itself when the 50-mgal contour does not pass exactly through a 50 mgal observation. A reduction in grid spacing would remove this problem.

Line data

A more difficult set of data to contour is one whose density of sampling is not isotropic. The data for the total intensity aeromagnetic maps of Figure 2 and Figure 3 were taken at 0.8-km intervals along flight-lines nominally 3.2 km apart and at a height of 650 m above ground. The grid spacing used in the contouring was 0.5 km and the number of iterations was 60. The number of grid points is 6000 in Figure 2 and 16,500 in Figure 3.

The general smoothness is satisfactory although a relatively rough profile along a line has an effect on the contours close to the flight-line, and this may not be desirable. The interpolation is not a product of one-dimensional splines. A section across the flight-lines is not the result of a one-dimensional spline. This may be a drawback in some cases, but enables trends not lying at right-angles to the flight path to be displayed.

CONCLUSION

The principle of minimum total curvature provides a method of two-dimensional interpolation which allows a computer to draw reasonable maps of geophysical data. The results are not always as a draughtsman would have them, but are an adequate substitute in most cases.

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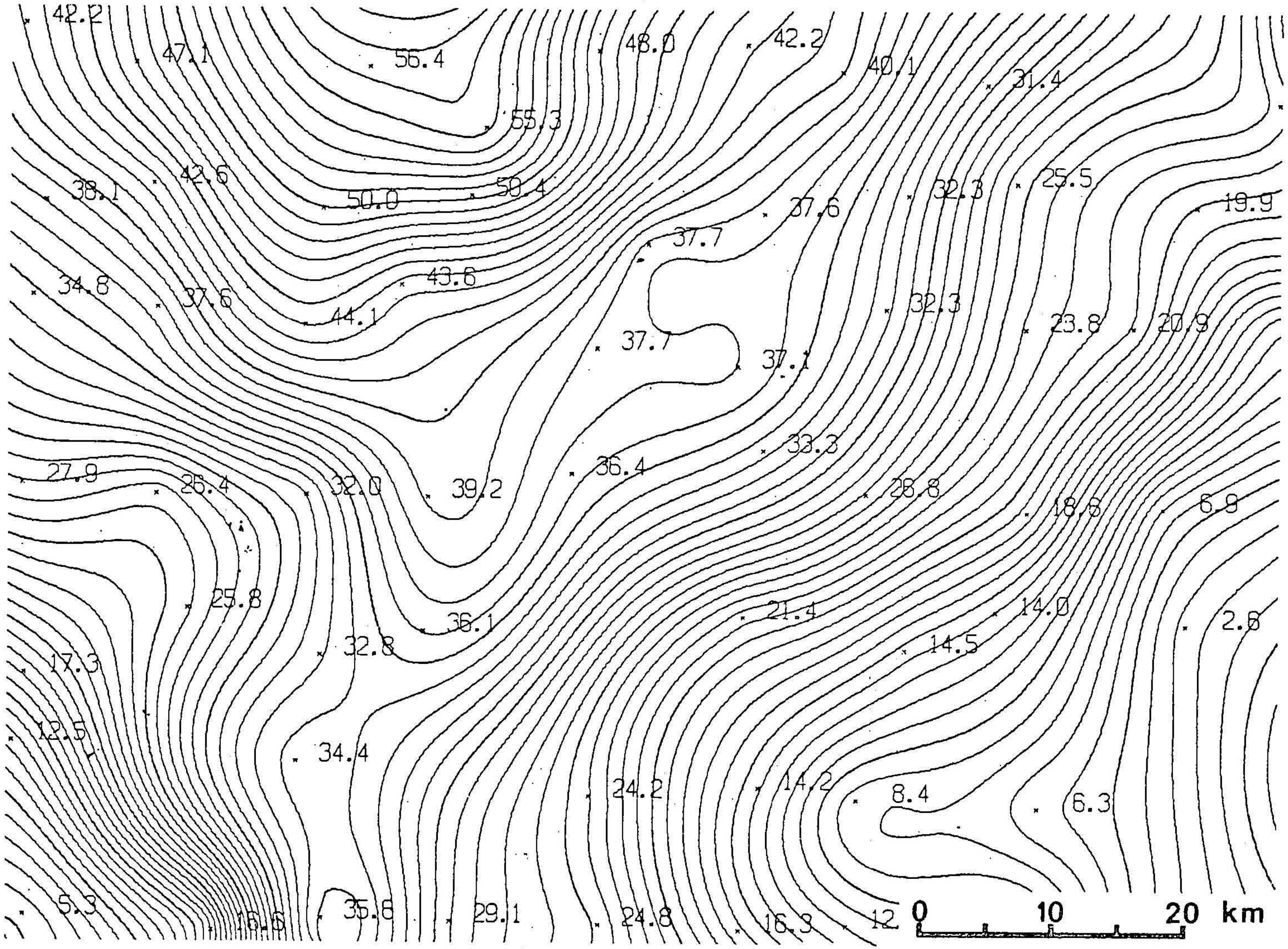

Fig. 1 Gravity data contoured at 1-mgal intervals using a grid spacing of 1.85 km

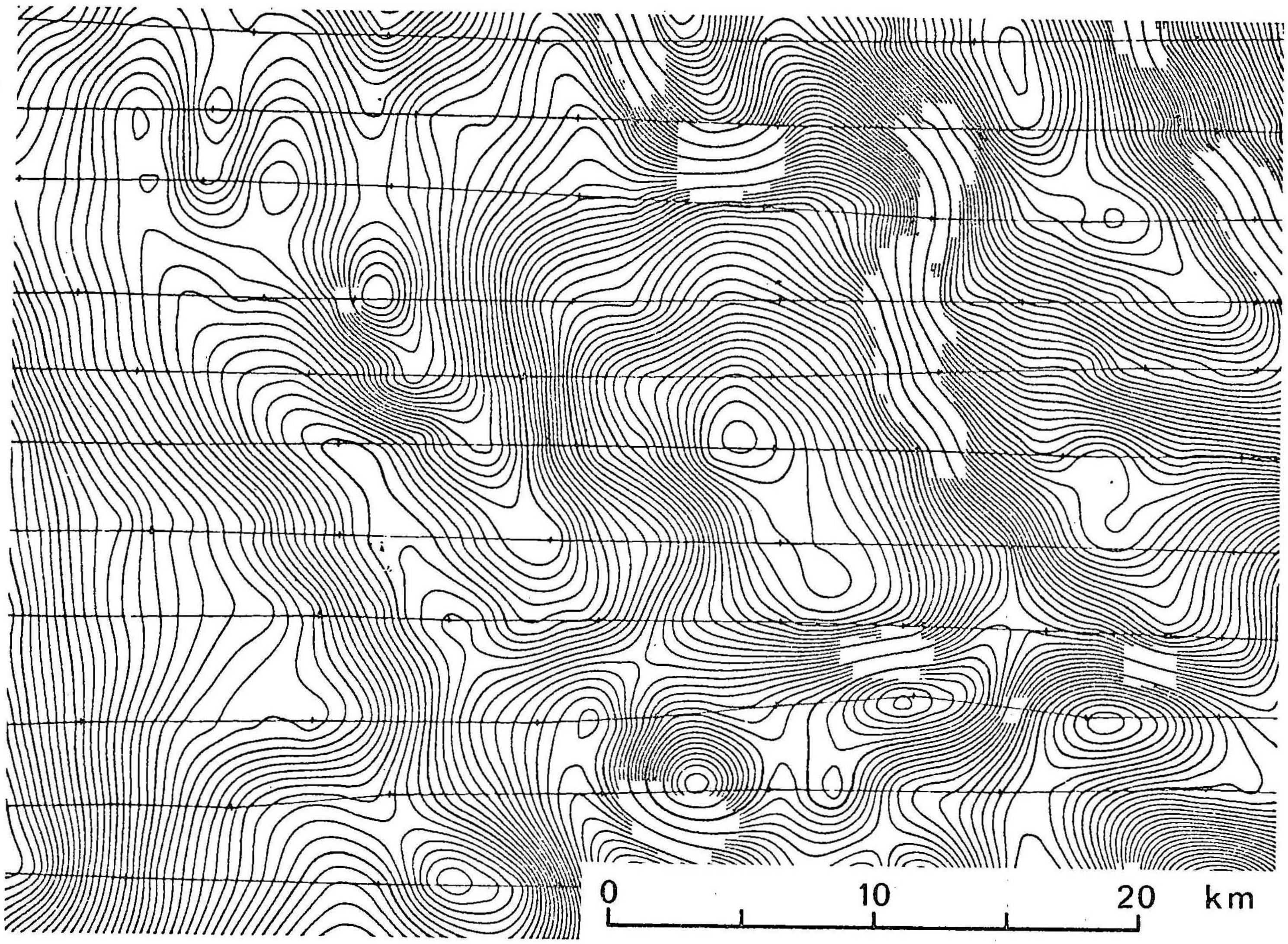

Fig. 2. Aeromagnetic data contoured at 10-gamma intervals using a grid spacing of 0.5 km

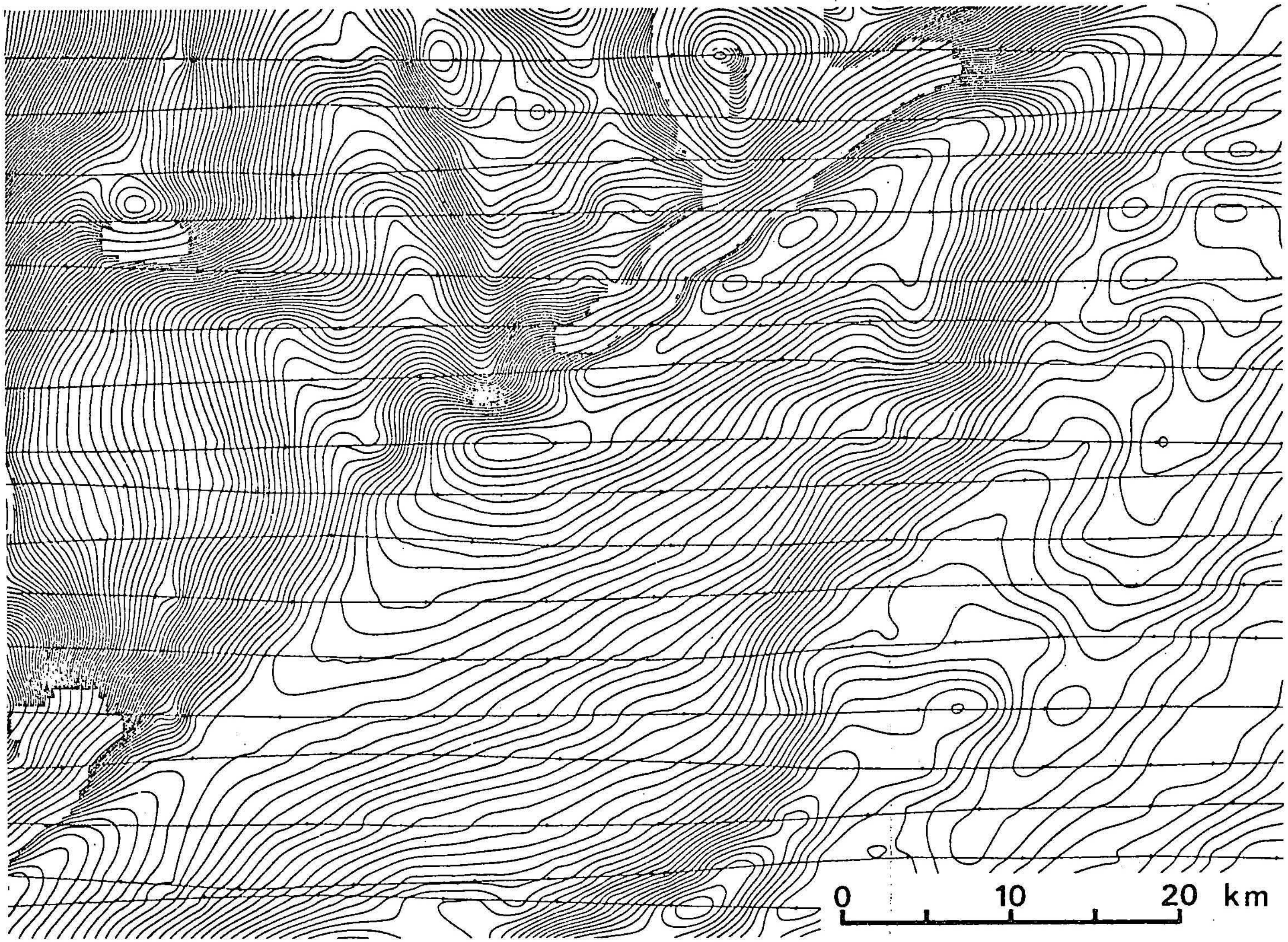

Fig. 3. Aeromagnetic data contoured at 10-gamma intervals using a grid spacing of 0.5 km