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TWO-DIMENSIONAL INTERPOLATION OF IRREGULARLY SPACED DATA USING POLYNOMIAL

by

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Two-dimensional interpolation of irregularly spaced data using polynomial splines

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ABSTRACT

A method has been developed for interpolating field data specified at irregularly spaced points over an area, such as is often the case with geophysical observations. The area is divided into triangles, of which the data points form the vertices. The interpolating function consists of a polynomial within each triangle. Across the boundary of each adjacent pair of triangles, the values of the polynomials and their derivatives to a prescribed degree are matched. Third degree polynomials permit matching of first derivatives, and fifth degree polynomials are required for matching second derivatives. Boundary conditions may be imposed on the derivatives, and equations may be introduced to improve the smoothness of the interpolating function.

The analytical nature of the function permits certain manipulations, for example calculation of directions of rays and intersection points in refraction seismology. The function could also be used for automatic contouring.

INTRODUCTION

Since about 1960, polynomial spline functions have been used successfully for interpolation of data specified at points spaced regularly or irregularly along a line (see e.g. Ahlberg, Nilson and Walsh, 1967). The most commonly used form consists of a different cubic polynomial in each interval with continuous first and second derivatives at the data points or nodes; the third derivative may be discontinuous at the nodes.

In extending splines to two dimensions, most attention has been devoted to data specified at points forming a rectangular grid (e.g. Ahlberg et al, loc cit; Bhattacharya, 1969). Bicubic polynomials were used in each rectangle; first derivatives, and some but not all second derivatives, were continuous at the nodes and edges of the rectangles. Hessing et al (1972) extended bicubic interpolation to irregularly spaced data by drawing lines through the data points to form quadrangles; extra points are needed at some intersections of these lines in order to complete the quadrangles, and values were calculated from neighbouring data for these points before beginning the interpolation.

Briggs (1974) developed a method for contouring irregularly spaced data, based on minimizing the curvature $\int (\nabla^2 S)^2$ of the spline function; this is equivalent to putting $\nabla^4 S = 0$ everywhere except at the nodes.

Any finite set of points in the plane can be connected by a mesh of which each element is a triangle; the points form the vertices of these triangles, and no additional intersecting points are introduced by crossing edges. This suggests interpolation by a spline function defined as a polynomial in each triangle. Hall (1969) divided a rectangular mesh into triangles by diagonals, and showed that bicubic polynomials could not be used for a second order spline functions on this mesh. Splines based on a triangular mesh have been used for approximating the solutions of certain differential equations, e.g. by Birkhoff and Mansfield (1974), who give references to previous work. Some of these splines use certain rational functions as well as tricubic polynomials (i.e. a polynomial which is cubic on the edges of the triangle).

In the following, we investigate the use of polynomial spline functions on a triangular mesh to interpolate irregularly spaced data, giving globally continuous derivatives to first or second order.

STATEMENT OF THE PROBLEM

Let (x, y) be coordinates in a Cartesian system in the plane. Let $\{x_i, y_i : i = 1, \ldots V\}$ be data points, at each of which a number g_i is specified. We require a function $S(x, y)$ with continuous derivatives to order 1 (i.e. $S \in \mathcal{C}^1$) such that

$$S(x_i, y_i) = g_i, \; i = 1, \ldots V \qquad (1)$$

We seek a solution by joining pairs of the V points by E non-intersecting line intervals which form the edges of T triangles. In each triangle T_j $(j = 1, \ldots T)$, we put

$$S(x, y) = P_j\,(x,y) \qquad (2)$$

where P_j is a polynomial in x and y of order n.

We bear in mind the following requirements:

(1) The function $S(x,y)$ should have at least as many parameters (i.e. coefficients of polynomials) as there are relations to be satisfied in order to ensure continuity of derivatives to order 1.

(2) Any excess of parameters over relations will enable boundary conditions to be imposed, and/or smoothness conditions to be satisfied (i.e. minimization of certain second or higher derivatives, or of functions or combinations of these).

(3) The order of the polynomial should not be too high, as this may permit unwanted oscillations between the data points.

(4) $S(x,y)$ should be of a uniform character, e.g. independent of azimuth or choice of co-ordinates axes as far as possible, and using the same order of polynomial for each P_i.

Most attention to date has been given to bicubic polynomials - i.e. polynomials of the form

$$P_i\ (x,y) = \sum a_{jk}\ x^j y^k,\ 0 \leqslant (j,k) \leqslant 3 \qquad (3)$$

These reduce to cubic polynomials along lines parallel to the x or y axis, but in general are 6th order polynomials in a variable u along a direction oblique to these axes. Thus there is an inherent difference between directions parallel to the co-ordinate axes and other directions.

We consider here two-dimensional polynomials of order n, i.e. of form (3), but with $0 \leqslant j + k \leqslant n$ without other restriction (which have (n+1) (n+2)/2 parameters each), and two dimensional polynomials of order n restricted to polynomials of order $m<n$ along the sides of all triangles. This restriction reduces the number of free parameters N_p in each triangle by 3(n-m) so that:

$$N_p = \tfrac{1}{2}\ (n+1)\ (n+2) - 3(n-m) \qquad (4)$$

We also consider polynomials satisfying $\nabla^4 S = 0$ inside each triangle; these must have n = 4 with the additional restriction

$$\nabla^4\ P_i\ (x,y) = 24a_{40} + 8a_{22} + 24a_{04} = 0$$

For these, $N_p = 14$.

<u>Homogeneous co-ordinates</u>

It is convenient to use homogeneous triangular co-ordinates z_1, z_2, z_3, defined as the perpendicular distances of a point from the three edges of a triangle (fig. 1).

Let the triangle $A_1A_2A_3$ (fig. 1)be numbered sequentially in a clockwise sense, with edges of length p_i opposite vertex A_i. Let q_i be the length of the perpendicular from A_i to the opposite edge, and α_i be the internal angle at A_i, taken as positive. Let R be the point (z_1, z_2, z_3). Any two of these are sufficient to specify R; they satisfy

$$\sum_{i=1}^{3} z_i/q_i = 1 \qquad (5)$$

Let the edge A_iA_j of the triangle make an angle θ_k, where $k = 6-(i+j)$, clockwise from the y axis. Then the triangular co-ordinates of a point (x, y) are derived from

$$z_k = (x-x_i) \cos \theta_k - (y-y_i) \sin \theta_k \qquad (6)$$

where x_i, y_i are the co-ordinates of the vertex A_i. z_k is positive for points inside the triangle if $A_i\ A_j$ is in a clockwise direction.

We assume that the triangle is not degenerate, i.e. no two sides are collinear or coincident.

The polynomial $\sum a_{mk}\, x^m y^k$ $(n \geq (m+k) \geq 0)$ can be transformed to a homogeneous form:

$$P\ (z_1,\ z_1,\ z_3) = \sum_{l+m+k\ =\ n} c_{lmk}\, z_1^l\ z_2^m\ z_3^k \qquad (7)$$

There are the same number of coefficients c_{lmk} as there were a_{mk} in the original polynomial. The order of the polynomial will remain the same for any such transformation.

This symmetrical form has the advantage that the polynomial can be evaluated at any vertex $(z_i = q_i,\ z_j = z_k = 0)$, or along any side of the triangle $(z_i = 0)$. Also the condition that a point lies inside the triangle is simply $z_i > 0$, $i = 1, 2, 3$.

Now let B_i be a point in the side $A_j\ A_k$ opposite vertex A_i (see fig. 2, where i = 3), and let u_i be the distance $A_j\ B_i$ where $j = i \bmod 3 + 1$.

Let v_i be the distance $A_k\ B_i$, so that $u_i + v_i = p_i$. Then for B_i we have $z_i = 0$, and hence

$$z_k = u_i \sin \alpha_j \qquad (8)$$

$$z_j = v_i \sin \alpha_k$$

By these transformations, e.g. $P(z_1, z_2, z_3)/_{z_3 = 0}$ can be expressed in the form

$$Q(u_3, v_3) = \sum_{m+l = n} b_{lm} u_3^l v_3^m \qquad (9)$$

where

$$b_{lm} = c_{lm0} (\operatorname{cosec} \alpha_2)^l (\operatorname{cosec} \alpha_1)^m \qquad (10)$$

with similar expressions for the polynomials along the other sides of the triangle.

EQUATIONS FOR MATCHING DERIVATIVES

Zero Order

For each triangle, three equations are derived from (1), e.g.

$$P(q_1, 0,0) = c_{n00} q_1^n = g_i$$

and hence

$$c_{n00} = g_i/q_1^n \qquad (11)$$

with similar expressions for c_{0n0}, c_{00n}.

For continuity across the common edge of two triangles, two polynomials Q, Q' of the form (9) must be equated for all $u = v'$, $v = u'$ along the edge (fig. 2). This involves $m + 1$ coefficients, but two relations exist between them by virtue of (11) at the end points; hence $m - 1$ additional equations are introduced for each common edge (where m is the degree of the edge polynomials: $m \leqslant n$).

First Order

The gradient at vertex V_j can be represented by two components $G_{jx} = dS/dx$, $G_{jy} = dS/dy$, which are taken as unknown. For each edge meeting at V_j, there is an equation of the form.

$$\left.\frac{dQ}{du}\right|_{u = 0 \text{ or } p} = G_{jx} \cos \theta + G_{jy} \sin \theta \qquad (12)$$

For continuity, G_{jx} and G_{jy} must have the same values for all edges meeting at V_j. Since each edge joins 2 vertices, there will be 2E such equations. Elimination of the 2V unknown quantities (G_{jx}, G_{jy}: $j = 1, \ldots, V$) results in 2(E-V) relations between the coefficients of the polynomials $\{Q\}$, and hence of $\{P\}$.

As $Q = Q'$, derivatives of all orders with respect to u (or v) are also equal across common edges. If z_3 and z_3' are the coordinates perpendicular to the edge (Fig. 2), then we need

$$\frac{dP}{dz_3} = \frac{-dP'}{dz_3'} \qquad (13)$$

Now $\left.\frac{dP}{dz_3}\right|_{z_3=0}$ can be expressed as a polynomial in (u, v) of order n-1. Thus n relations between the two sets of coefficients are needed, but as the gradients have already been matched by (12) at the end points, n-2 new relations are required for each common edge.

Thus for matching first derivatives, we have:

3T equations of type (11);

(m-1) E_a equations from $Q = Q'$, where E_a is the number of internal edges;

2(E-V) equations derived from (12)

$(n-2)E_a$ equations derived from (13)

The total number is then

$$D_1 = 3T - 2V_a + (m+n-1)E_a \qquad (14)$$

Second Order

The second order derivatives or curvature is specified by three parameters $\frac{d^2S}{dx^2}$, $\frac{d^2S}{dy^2}$, $\frac{d^2S}{dxdy}$. By arguments similar to those for the gradients, and noting that derivatives of the form $d^2P/dzdu$ are already equated across each side, we need 2E - 3V equations to match the second derivatives at the vertices, and n-3 equations across each internal edge. This gives a total of

$$D_2 = 3T-4V_a-V+(m+2n-2)E_a \qquad (15)$$

TRIANGULATION OF A SET OF POINTS

One of the main requirements is to select S(x,y) so that it has sufficient parameters to enable the required derivatives to be matched at vertices and edges; thus it is important to know the numbers of triangles and edges in a triangulation of a set of V points.

Any network of polygons enclosed in and covering a bounded polygon in the plane satisfies Euler's equation for a "sphere with one boundary" in the language of topology:

$$F+V = E+1 \qquad (16)$$

where F = ~~no.~~ number of polygons (non-overlapping)

V = ~~no.~~ number of vertices

E = ~~no.~~ number of edges.

For a triangulated set of points, let the outer boundary form a polygon with E_b sides and $V_b = E_b$ vertices. Then we have, in addition to (16), by replacing F ~~by~~ with T (the number of triangles):

$$3T = 2E-E_b = E+E_a = 2E_a+E_b \qquad (17)$$

where the suffix a is used to denote interior edges or vertices; and from (16) and (17) we can deduce further relations:

$$T = V+V_a-2 \qquad (18)$$

$$E = 2V+V_a-3 \qquad (19)$$

From (18) and (19) it is clear that, for a given set of V points, once the number V_b of points in the boundary has been determined, then the total numbers of triangles and edges are known.

PARAMETERS AND EQUATIONS

The numbers of parameters N_p determined from (4) for various polynomials are given in Table 1. Also listed are the numbers of equations D_1 and D_2 required for matching from (14) and (15). These have been transformed by (16), (17), (18), and/or (19) into a form suitable for comparison with N_p.

The lowest order polynomial for which first derivatives can be matched for any set of points is of order 3 with no reduction in order along the sides. This leaves $V+V_b+1$ degrees of freedom which can be used for boundary or smoothing conditions, e.g. one boundary condition for each boundary edge, plus one relation at each vertex, plus one additional general relation, or 2 boundary conditions for each edge, plus one relation for each internal vertex, plus one additional general relation.

For matching second derivatives the lowest order polynomial is $n = 5$, $m = 5$. This permits four boundary conditions for each boundary edge, plus two relations for every vertex, or some equivalent combination of relations.

Other polynomials, such as $n = 4$, $m = 2$ for first derivatives, and $n = 5$, $m = 4$ for second derivatives, could be used for some but not all sets of points, as they depend on the relative numbers of internal and boundary points. These also have disadvantages with respect to the simple third and fifth order polynomials as regards conditions (4) and (2) above.

It is of interest however, that second derivatives could be matched by a cubic polynomial if

$$2V_a - V_b - 7 < 0$$

i.e. $V_b > 2V_a - 7$, or $V_b > (2V-7)/3$

This condition is satisfied by a square array of n^2 points if $n \leq 5$.

CUBIC POLYNOMIALS

It was decided in the first place to study third-order polynomials, matching first derivatives; such a surface should be satisfactory e.g. for representation of a seismic refractor for 2-dimensional ray-tracing or similar computations. We show how to reduce the problem from 10T unknown parameters to 2V parameters and the same number of equations.

We have three equations of type (11), which give c_{300}, c_{030}, and c_{003} directly in terms of g_j; thus we can reduce the number of unknown parameters from 10T to 7T.

From (9), we ~~get~~ derive for the gradient at a vertex in the direction along an edge

$$\left.\frac{dQ}{du}\right|_{u=0} = p^2 \; (3b_{03}-b_{12}) \qquad (20)$$

b_{03} is related to c_{030} by (10); hence (12) expresses b_{12} in terms of G_x, G_y, and known quantities. For any three edges $E_i (i = 1,2,3)$ meeting at a common vertex V, let B_i denote b_{12} or b_{21} according as u = 0 or v = 0 at V. Then eliminating G_x and G_y from the three equations of type (12) gives

$$(p_1^2 B_1 - 3g/p_1) \sin(\theta_3 - \theta_2) + (p_2^2 B_2 - 3g/p_2) \sin(\theta_1 - \theta_3) + (p_3^2 B_3 - 3g/p_3) \sin(\theta_2 - \theta_1) = 0 \qquad (21)$$

For practical calculation, two edges most nearly at right angles are selected as reference lines, and an equation of form (21) is written for these two edges together with each other edge in turn meeting at that vertex. We define a vector $\underline{B}$ of dimension 2V, whose elements consist of the reference $\{B_i\}$.

From such equations at all vertices, the coefficients b_{12} and b_{21} in equation (9) can be determined for every edge in terms of $\underline{B}$ and the known quantities p, g and θ; thus we get 2E-2V equations of this nature.

The coefficients c_{012}, c_{021}, c_{102}, c_{201}, c_{120}, and c_{210} for each triangle can then be determined by (10) in terms of $\underline{B}$. As the data values and gradients are matched at both ends of the edge for the polynomials P and P' in the triangles on either side of the edge, it follows that the edge polynomials Q and Q' are matched along the edge (fig. 2).

The transverse gradient at an edge is given by expressions such as:

$$\left.\frac{dP}{dz_3}\right|_{z_3=0} = (c_{201} - c_{210}\cos\alpha_1 - 3c_{300}\cos\alpha_2)z_1^2 + (c_{111} - 2c_{120}\cos\alpha_1 - 2c_{210}\cos\alpha_2)z_1 z_2 + (c_{021} - c_{030}\cos\alpha_1 - c_{120}\cos\alpha_2)z_2^2 \qquad (22)$$

As the gradients are matched at the ends of each edge, (i.e. $z_1 = 0$; $z_2 = 0$) the coefficients of z_1^2 and z_2^2 are equal for P and P'. We write $B_a = b_{12} = b'_{21}$ and $B_b = b_{21} = b'_{12}$ and $C = c_{111}$; then matching the coefficients of $z_1 z_2$ and $z'_1 z'_2$ gives

$$C\sin\alpha_1\sin\alpha_2 + C'\sin\alpha'_1\sin\alpha'_2 = 2B_a(\cot\alpha_1 + \cot\alpha'_2) + 2B_b(\cot\alpha_2 + \cot\alpha'_1) \qquad (23)$$

As B_a and B_b can be expressed in terms of the vector $\underline{B}$, the RHS of (23) can be transformed to an expression involving elements of $\underline{B}$ as the only unknown parameters. Together with C_i, i = 1, ..., T, we now have 2V+T unknown parameters, with E_a relations between them, leaving $2V_b+V_a+1$ degrees of freedom.

Boundary conditions

The surplus degrees of freedom may be used to minimize the curvature of $S(x,y)$, i.e. $\int(\nabla^2 S)^2$ could be minimized, where the integral is taken over the area covered by the triangles. This involves use of Lagrangian parameters or generalized inverses, and increases the size of the problem substantially. This approach is discussed later.

Another approach is to devise V_a+2V_b+1 extra equations which would tend to reduce but not necessarily minimize the curvature; there are then exactly enough equations to determine the required parameters.

Following Briggs (op. cit.), it is desirable to have $\nabla^2 S = 0$ along the boundary of the polygon. As $\nabla^2 P_i$ is a linear function in the homogeneous co-ordinates (u,v) along any edge of a triangle, and hence along any boundary edge of the polygon, we can achieve this by putting $\nabla^2 P_i = 0$ at both ends of each boundary edge; this gives $2V_b$ equations, leaving a further V_a+1 to be found.

Now

$$\nabla^2 P = h_1 z_1 + h_2 z_2 + h_3 z_3 \qquad (24)$$

where

$$\left.\begin{aligned} h_1 &= 6c_{300}+2c_{120}+2c_{102}-2c_{111}\cos\alpha_1-4c_{201}\cos\alpha_2-4c_{210}\cos\alpha_3 \\ h_2 &= 6c_{030}+2c_{210}+2c_{012}-4c_{021}\cos\alpha_1-2c_{111}\cos\alpha_2-4c_{120}\cos\alpha_3 \\ h_3 &= 6c_{003}+2c_{201}+2c_{021}-4c_{012}\cos\alpha_1-4c_{102}\cos\alpha_2-2c_{111}\cos\alpha_3 \end{aligned}\right\} \qquad (25)$$

If the edge $z_1 = 0$ is a boundary edge, then the boundary conditions are

$$h_2 = h_3 = 0 \qquad (26)$$

It is possible that a boundary point is a vertex of only one triangle. In this case, putting $\nabla^2 P_i = 0$ at this vertex gives only one equation instead of the two normally obtained for a boundary vertex. To compensate for this, it is proposed to equate $\frac{d^2f}{du^2}$ $\left(\text{or } \frac{d^2f}{dv^2}\right)$ for the two edges where they meet at this point; this means that the principal axes of curvature will bisect the angles formed by these edges.

All the coefficients except c_{111} can be expressed in terms of $\underline{B}$; hence either $h_2 = 0$ or $h_3 = 0$ gives an expression for c_{111} in terms of $\underline{B}$; we also get an additional relation between elements of $\underline{B}$. The boundary edges give a total of V_b such relations, and reduce the number of unknowns by V_b. We can express the remaining C_i in terms of $\underline{B}$ by the following procedure.

We arrange the triangles in sets (fig. 3) defined inductively as follows:

Set 1 consists of triangles with an edge forming part of the boundary polygon.

For $k>1$, Set k consists of triangles which do not belong to any Set m $(m<k)$, but which have a common edge with a triangle in Set k-1. Then C_i for T_i in Set 1 have been expressed in terms of $\underline{B}$ by (26). C_j for T_j in Set k may be expressed in terms of $C_{j'}$, for some $T_{j'}$, in Set k-1 and elements of $\underline{B}$ by (23). Thus by working sequentially through sets, all C_j may ultimately be expressed in terms of $\underline{B}$.

If T_l and T_m are both in Set k and have a common edge, or if T_l has more than one common edge with triangles in Set k-1, (23) gives relations between C_l and C_m additional to those needed to express C_l and C_m in terms of $\underline{B}$, and these can be converted to equations between elements of $\underline{B}$.

We now have 2V unknown parameters and $V+V_b-1$ equations; we need another V_a+1 to make a square matrix.

The occurrence of V_a suggests deriving an extra relation at each internal vertex. There are three or more triangles meeting at each of these. Values of $\nabla^2 P_i$ for these triangles will be in general different at a given vertex. It would be desirable to make the sum of squares of these values a minimum, but a large number of parameters is involved and again Lagrangian multipliers or a similar technique would have to be used. On the principle that in minimizing a sum of squares of linear quantities, the first move is to make their average equal to zero, the approach adopted here is to equate the average (or the sum, which comes to the same thing) of the values of $\nabla^2 P_i$ at each internal vertex to zero.

We still need one equation. Somewhat arbitrarily, it was decided to equate the sum of values of $\nabla^2 P_i$ at all boundary vertices to zero. Since the values for boundary triangles (i.e. triangles of Set 1) are already zero, this means the sum of the values for triangles with just one vertex on the boundary.

CONCLUSIONS

It has been shown that a third degree polynomial is capable of matching first derivatives, and that the problem may be reduced to solving 2V equations for 2V parameters. By similar procedures, a fifth degree polynomial may be used for matching second derivatives; the number of equations and parameters for this case is 5V.

An alternative approach is to minimize the integrated least square total curvature or differential curvature. Also, different boundary conditions may be applied, e.g. $d^2S/dz^2 = 0$ across boundary edges instead of (26).

[redacted]

The method is applicable where a smooth interpolant is required which can be calculated readily at intermediate points. The function S(x,y) could be used for drawing contours, but it has not yet been demonstrated whether it would have any advantage over other methods such as that of Briggs (1974).

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<u>TABLE 1</u>

<u>Parameters and equations for matching first and second order derivatives</u>

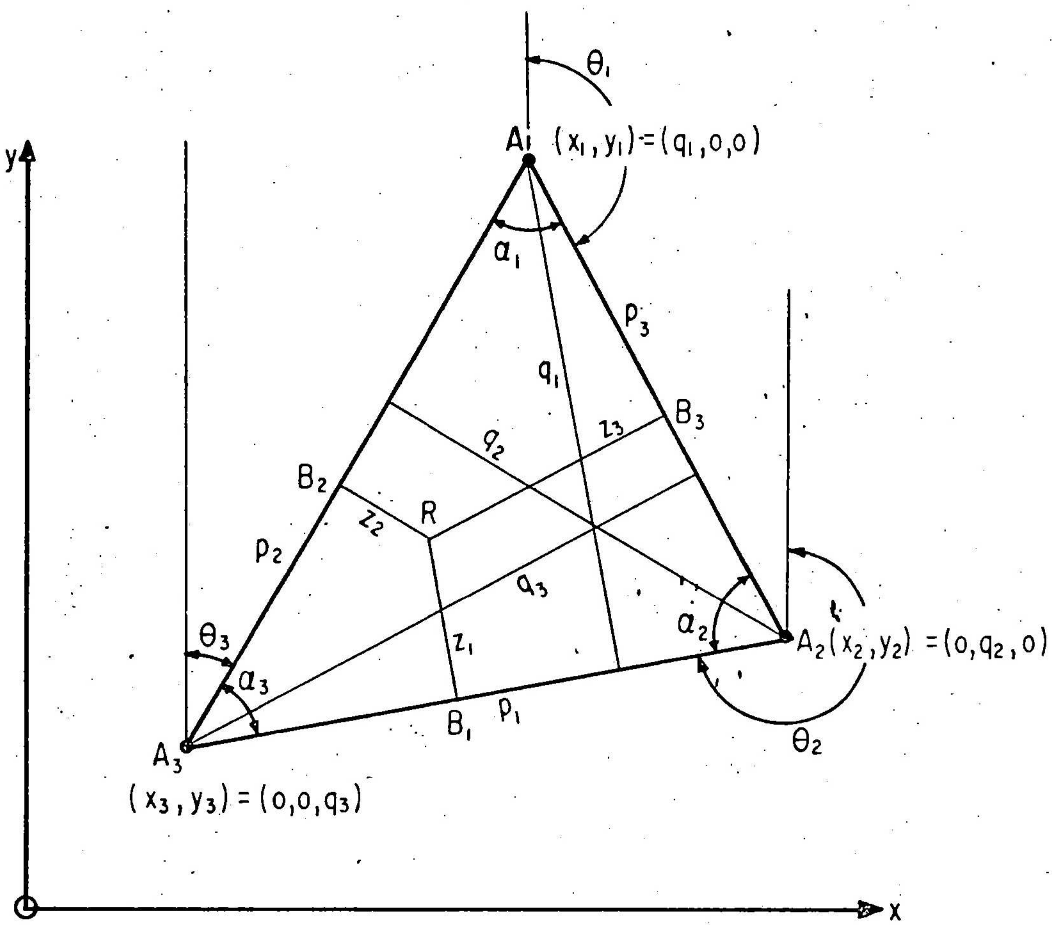
n	m	N_p	D_1	$N_p - D_1$			D_2	$N_p - D_2$		
2	2	6T	$5T-2+1E_a$	$-1V_a-3$						
3	3	10	$5T-2+3E_a$	$2V_b+1V_a+1$			$7T-V-4+3E_a$	$1V_b-2V_a+7$		
3	2	7	2	0	−2	+4	2	−1	−5	+10
4	4	15	5	5	+5	−3	6	3	−1	+6
4	3	12	4	3	+2	0	5	1	−4	+9
4	2	9	3	1	−1	+3	4	−1	−7	+12
4	$\nabla^4 P = 0$	14	5	4	+3	+3	6	2	−3	+8
5	5	21	7	9	+11	−9	9	6	+2	+3
5	4	18	6	7	+8	−6	8	4	−1	+6
5	3	15	5	5	+5	−3	7	2	−4	+9
6	6	28	9	14	19	−17	12	10	+7	−2
6	5	25	8	12	16	−14	11	8	+4	+1
6	4	22	7	10	13	−11	10	6	+1	+4
6	3	19	6	8	10	−8	9	4	−2	+7

Figure Captions

Figure 1. Homogeneous triangular co-ordinates

Figure 2. Relations across the common boundary of two adjacent triangles.

Figure 3. Ordering of triangles in sets to facilitate reduction of equations for coefficients C.



y
x
O
θ_1
A_1 $(x_1, y_1) = (q_1, 0, 0)$
α_1
p_3
q_1
B_3
z_3
q_2
B_2
R
z_2
p_2
q_3
α_2
$A_2 (x_2, y_2) = (0, q_2, 0)$
θ_3
α_3
z_1
B_1
p_1
θ_2
A_3
$(x_3, y_3) = (0, 0, q_3)$

Fig. 1

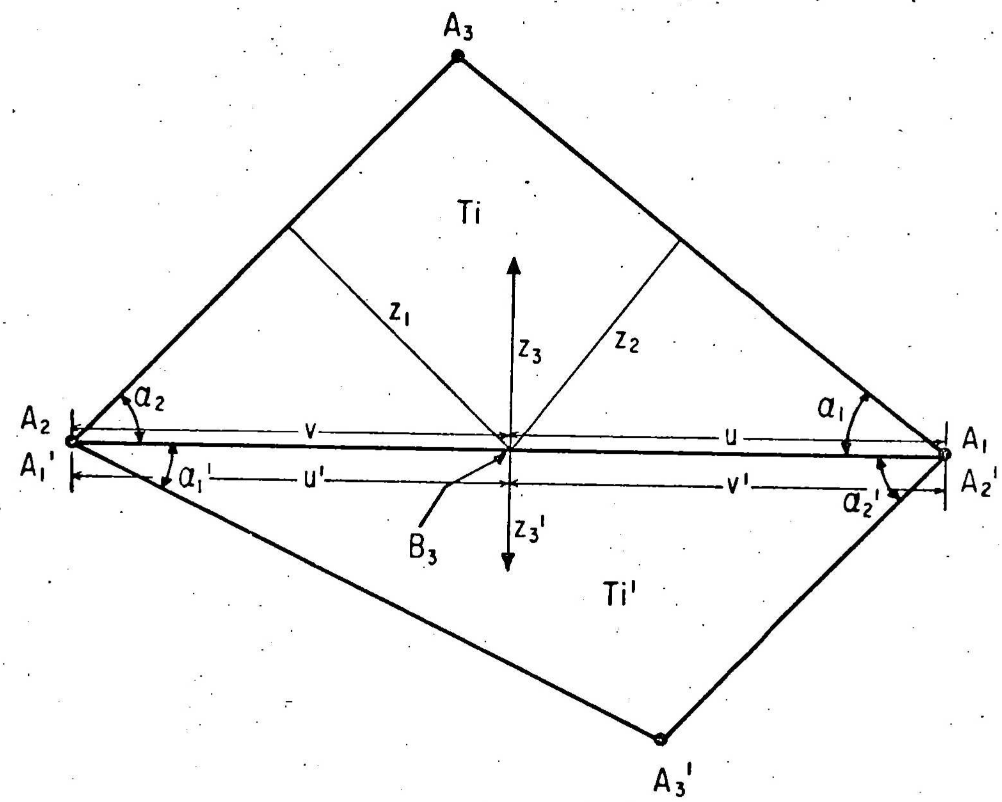


Fig. 2

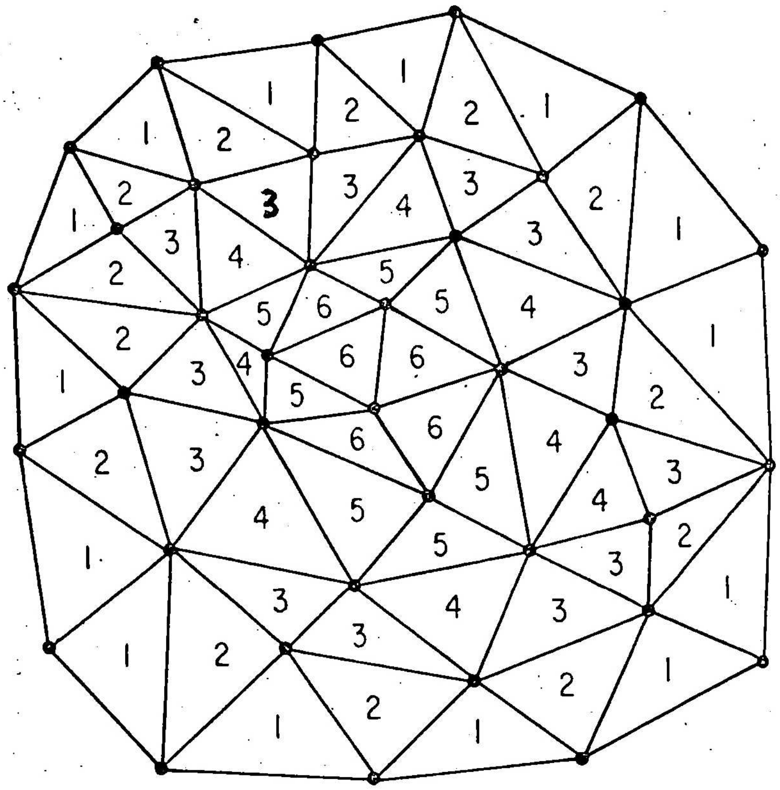


Fig. 3