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Abstract

The temperature function T for a body of thermal conductivity k in a layered medium of thermal conductivity k^* satisfies the integral equation

$$T = T^* + \frac{M}{2\pi}\int_s \frac{\partial T}{\partial n} G\,ds$$

Here T^* is the temperature for the region without the heterogeneity, G the Green's function and M is a constant $(k^*-k)/(k^*+k)$ and $\partial T/\partial n$ is the normal derivative, in the direction of the outward normal to the surface S, and is evaluated on the surface S.

For irregular shaped bodies in layered structures, the integral is replaced by a series of surface integrals, each of which is planar over a triangular facet. Together the facets help to define the surface of the body. The integral equation is solved by assuming that $\partial T/\partial n$ is constant over each facet and by forming a set of simultaneous equations for the various $\partial T/\partial n$ by applying the boundary condition for the normal flow of heat across each facet.

1. *Introduction*

Surface heatflow, when interpreted in conjunction with other geological and geophysical information, is an important constraint for theories regarding the history of the earth. Heatflow anomalies are caused by different geological processes, including tectonic movement and volcanic activity, and these are particularly evident on rifted margins where new sedimentary basins are forming. Study of the present day localised heatflow often enables paleo-heatflow to be modified. This information greatly improves the understanding in the thermal and burial history of sedimentary basins and their hydrocarbon potential.

Local heatflow anomalies may be attributed to contrasts in thermal conductivity and the resulting refraction of heat in subsurface sediments, variations in sedimentation rate or erosion, topographic variations with their resulting uneven sediment distribution, and finally convection of ground water. Other local factors that may affect the heatflow results are oxidation of ore bodies and local concentrations of radioactive materials.

In the steady state heatflow analysis in the oceanic basins, the simple relationship of heatflow influx as a linear product of vertical thermal gradient and thermal conductivity is valid only when the flux of heat is one dimensional, i.e. the flux vector is normal to the planar cooling surface. This ideal situation would only exist in homogeneous media, and thermal conductivity variations if any must be bound by sediment layers with planar surface parallel to the ocean floor, or normal to the heat flux.

i. Two dimensional heterogeneity

In general, the assertion that the bounding surface is planar only exists in an ideal theoretical model. Within a typical sedimentary basin, geological heterogeneity and irregular complex topographic features are commonplace, resulting in the refraction of heatflow across near surface boundaries.

Therefore the steady state heatflow model across inhomogeneous and irregular layer structures are generally at least two dimensional, with discontinuous temperature flux across boundaries of dissimilar conductive media. The extent of any perturbation in the oceanic surface heatflow measurement generally depends on the conductivity ratio and boundary configuration.

The refraction of heatflow across dissimilar conductive media and irregular topography has been studied by many authors. One of the earliest analytical studies on semi-elliptical cylinders was by Lachenbruch and Marshall (1966). Sclater and Miller (1969) described a numerical method using the relaxation technique for computing the effect of two layered conductivity structures, and this was later extended by Sclater, Jones & Miller (1970) for more complex two dimensional structures. Geertsma (1971) applied a finite element technique for temperature profiles over a salt dome and faults. Lee and Hentley (1974) studied heat refraction by cylindrical structures. England (1976) and Francis & Wheildon (1978) employed finite element methods to analyse the effect of heatflow refraction due to intruded bodies into a sedimentary sequence.

Most of the workers cited above based their analyses on the calculation of surface heatflow effect on artificial models, with little reference to the possible perturbation of the whole heatflow field due to local topography in actual basin studies. Choi, Liu & Cull (1988) have applied relaxation techniques to analyse the heatflow measurement results of a heatflow cruise in the Queensland Trough, Western Coral Sea in Northeastern Australia. The steady state geothermal model used in their numerical analysis is closely related to the topographic structures of the Queensland Basin. Excellent agreement was achieved between the analytical results and the actual measured surface heatflow perturbation. Although refinement may be required for transient geothermal models to account for a limited thermal source, age and mantle convection, the steady state conduction model is sufficient to explain the systematic trends of reduced heatflow values towards the basin centre and perturbation due to heat refraction near the basin margins. They concluded that correlation between heatflow and sediment thickness can be used as a criterion to determine the nature of any large-scale heatflow perturbation in basin columns caused by non-conductive heat transfer mechanisms.

ii. Three dimensional heterogeneity

Simmons (1967), in his analysis of heatflow anomalies due to contrasts in heat production, dealt extensively with the analogous relationship between heatflow, gravity and magnetic anomalies based on potential theory. The solutions to surface heatflow anomalies due to six common geometrical shapes of buried heatflow source were readily found from the classical solutions to the Laplace or Poisson equations of gravitational and electric potential theory, depending upon whether sources of heat were present. His assumption of the boundary condition of zero temperature at the earth's surface is fairly accurate and valid in oceanic basins. His derivation was further bound by the assumption that thermal diffusivity and thermal conductivity are uniform and constant throughout the half space beneath the Earth's surface.

In reality the basin topographies are mostly in a layered heterogeneous structure in three dimensions with most rapid variation of properties in the vertical direction. Our present study is centred on the conduction of heat and the resultant surface anomalies due to the intruded bodies of arbitrary shape in a three dimensional heterogeneous environment. Here we extend the above methods of modelling the temperature distribution about a heterogeneity of thermal conductivity k in a layered ground by means of an integral equation. For this study the thermal diffusivity and the thermal conductivity are allowed to vary in the ground. Thus we are able to allow for the refraction of heat across dissimilar media of more general geological models than were considered by Lee & Henyey (1974). In the next section of the paper an integral equation will be derived which describes the temperature distribution about an arbitrarily shaped three dimensional structure, of thermal conductivity k , which is heated by means of constant temperature being maintained at the upper and lower boundaries of a layer or layers of conductivity k^*. The following section derives the Green's function or Kernel for the integral equation and shows that this expression may be transformed to a more convenient one for a thick layer. Finally the nature of the more general solution is investigated for the particular case of a spherical conductor. The purpose of the example is firstly to formulate the ideas which were introduced in the first part of the paper, and secondly to provide a basis for checking any computer program which is based on the results of the more general case.

2. Derivation of the Integral Equation for Steady State Heatflow

Let $\underset{\sim}{\Phi}$ denote the flux of heat, T the temperature and k the non homogeneous thermal conductivity. Then from the equation of heatflow

$$\underset{\sim}{\Phi} = -k \ \nabla T, \tag{1}$$

and

$$\nabla . \underset{\sim}{\Phi} = H, \tag{2}$$

Here H denotes the heat source density.

Combining equations (1) and (2) one finds that:

$$k\nabla^2 T = -\nabla k . \nabla T + H. \tag{3}$$

If T^* is the temperature of the ground without the conductor and k^* the corresponding value of the conductivity, then:

$$\nabla^2(T-T^*) + \frac{\nabla(T-T^*).\nabla k^*}{k^*} = 0, \tag{4}$$

outside the heterogeneity, and

$$\nabla^2(T-T^*) + \frac{\nabla(T-T^*).\nabla k^*}{k^*} = -\frac{\nabla T.\nabla k}{k} - \frac{\nabla k^*.\nabla T}{k^*} \tag{5}$$

inside the heterogeneity.

Let G the Green's function be a solution of the following equation and subject to the same boundary conditions as $T-T^*$

$$\nabla^2 G + \nabla G . \frac{\nabla k^*}{k^*} = -\delta(x - x')\delta(y - y')\delta(z - z') \tag{6}$$

Here $\delta\left(\underset{\sim}{x} - \underset{\sim}{x}'\right)$ is the Dirac delta function.

The point (x',y',z') lies within the heterogeneity. We proceed to derive an integral equation. Multiplying equations (4) and (5) by G and equation (6) by $(T\text{-}T^*)$, then subtract equation (6) from equations (4) and (5) and integrate over the region $z > o$. We obtain,

$$T = T^* + \int_V G \nabla T . \frac{\nabla k}{k} dv. \tag{7}$$

Here we have assumed that k is constant over the region occupied by the heterogeneity and V denotes the volume of the heterogeneity. After writing $k = (k^* - k)\underset{\sim}{H}(n)$, where $\underset{\sim}{H}(n)$ is the Heaviside step function and n is a co-ordinate in the direction of the outward normal one finds that,

$$T = T^* + 2M \int_S \frac{\partial T}{\partial n} G ds. \tag{8}$$

The procedure is analogous to that used by Lee (1972).

In equation (8), S is the surface of the heterogeneity, $M = (k^* - k)/(k^* + k)$, k is the conductivity of the heterogeneity and the derivative $\partial/\partial n$ denotes the derivative with respect to the outward normal n of the surface S and is evaluated on the surface of the heterogeneity.

3. The Green's Function

The simplest geometry is the case of a slab of thickness b with fixed temperatures at its boundaries. Since $T\text{-}T^*$ is zero at the boundaries and k is a constant, equation (6) reduces to

$$\nabla^2 G = -\delta(x - x')\delta(y - y')\delta(z - z') \tag{9}$$

We note T-T^* is zero at both the top where $z=b$ and at the base of the slab where $z=0$. So we set

$$\delta(z-z')=\frac{1}{b}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi z}{b}\right)\sin\left(\frac{m\pi z'}{b}\right) \tag{10}$$

and look for a solution of equation (9) in the form

$$G=\sum_{m=1}^{\infty}g_m(x,y)\sin\left(\frac{m\pi z}{b}\right)\sin\left(\frac{m\pi z'}{b}\right) \tag{11}$$

Substituting equations (10) and (11) into equation (9), taking the two dimensional Fourier transform with respect to x and y, and using the orthogonality properties of the sine functions yields,

$$\hat{g}_m\left(l^2+n^2+\frac{m^2\pi^2}{b^2}\right)=\frac{1}{b}e^{ilx'+iny'} \tag{12}$$

Here l and n are the wave lengths and transform variables of the Fourier transforms with respect to x and y respectively. The function $\hat{g}_m$ is defined by

$$\hat{g}_m=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x,y)g_m(x,y)dxdy \tag{13}$$

$$F(x,y)=e^{ilx+iny}$$

and

$$g_m=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F(x',y')F(-x,-y)}{b\left(n^2+l^2+\frac{m^2\pi^2}{b^2}\right)}dldn,$$

$$=\frac{1}{2\pi}\int_0^{\infty}\frac{pJ_0(pr)}{b\left(p^2+\left(\frac{m\pi}{b}\right)^2\right)}dp \tag{14},$$

where $p^2=n^2+l^2$.

and

$$r = \sqrt{(x - x')^2 + (y - y')^2}, \tag{15}$$

which simplifies to,

$$g_m = \frac{1}{2\pi b} K_0\left(\frac{m\pi}{b} r\right), \tag{16}$$

see Oberhettinger (1972, eq. 4.13).

Therefore

$$G = \frac{1}{2\pi b} \sum_{m=1}^{\infty} K_0\left(\frac{m\pi r}{b}\right) \sin\left(\frac{m\pi z}{b}\right) \sin\left(\frac{m\pi z'}{b}\right) \tag{17}$$

Equation (17) may be transformed to a useful result by using the result from Oberhettinger (1973, p.54, no.3.29).

Let $B(q) = \dfrac{1}{\sqrt{r^2 + (2bn + q)^2}}$

One finds that:

$$G = \frac{1}{4\pi} \left\{ \frac{1}{\sqrt{r^2 + z_-^2}} - \frac{1}{\sqrt{r^2 + z_+^2}} + \frac{1}{2} \sum_{n=1}^{\infty} \{ B(-z_-) - B(-z_+) + B(z_-) - B(z_+) \} \right\}. \tag{18}$$

Where $z_- = z - z'$ and $z_+ = z + z'$

.

For large b and with r and z fixed, equation (18) simplifies to,

$$G = \frac{1}{4\pi} \left\{ \frac{1}{\sqrt{r^2 + (z - z')^2}} - \frac{1}{\sqrt{r^2 + (z + z')^2}} \right\}. \tag{19}$$

Equations (17), (18) and (19) provide different ways of viewing the Green's function. The transformation of equation (17) to equation (18) shows that the harmonic series in equation (17) may be replaced by a series of images. Equation (19) shows that for large b only a single image about the plane $z=0$ suffices.

4. Example of a Spherical Conductor in a Layered Medium

Consider a spherical conductor of radius b and thermal conductivity k_2 in a ground of thermal conductivity k_1. This geometry can be described in terms of spherical polar co-ordinates (R,Θ,Ψ) or rectangular co-ordinates (x,y,z), the respective origins of which are at the centre of the sphere. The z axis is chosen to be positive in the vertical direction, with the ground surface being specified by $z=h$. It is assumed that were there no sphere the background temperature, T^*, would vary linearly between $z=h$ and $z=z_2$, below the sphere. If T^* equals T_1 at $z=h$ and $T^*=T_2$ at z_2 then:

$$T^* = Az + B,$$

where

$$A = (T_1 - T_2)/(h - z_2), \tag{20}$$

$$B = (T_2 h - T_1 z_2)(h - z_2).$$

We assume that $h-z_2$ is large relative to b, so that the Green's function of equation (19) may be used.

As there has been a shift in the origin of the co-ordinate system from the base of the layer to the centre of the sphere, equation (19) has to be modified to,

$$G = \frac{1}{4\pi}\left\{\frac{1}{\sqrt{r^2+(z-z')^2}} - \frac{1}{\sqrt{r^2+((z+z')-2z_2)^2}}\right\} \tag{21}$$

For this geometry the integral equation may be written:

$$T = T^* + \frac{M}{2\pi}\int_{sphere}\frac{\partial T}{\partial R}\left\{\frac{1}{P_1} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(z+z', x-x', y-y')\,d\alpha\, d\beta\right\}dV \tag{22}$$

where

$$Q(z,x,y) = \frac{e^{-\lambda z + i\beta x + i\beta y + 2\lambda z_2}}{\lambda},$$

where

$$M = \frac{(k_1 - k_2)}{(k_1 + k_2)}$$

$$P_1 = \sqrt{(x - x')^2 + (y - y')^2}$$

$$\lambda = \sqrt{\alpha^2 + \beta^2}$$

This integral equation may be solved by solving for the coefficients A_n in the expression for T given below

$$T = \sum_{n=0}^{\infty} A_n R_1^n P_n(\cos\theta) \tag{23}$$

Notice that the expression for T above, would be obtained if Laplace's equation were solved for T within the spherical conductor.

Substituting equation (23) into equation (22) and using result no 10 of table 1 of Lee (1975) and the orthogonality properties of the Legendre functions yields the result given in equation (24).

$$T = T_1^* + \frac{M}{2\pi} \sum_{n=0}^{\infty} \frac{2\pi n b^n}{2n+1} \{A_n P_n(\cos\theta) - \frac{b A_n}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-1)^n (\lambda b)^n \frac{Q(z,x,y)}{n!} d\alpha d\beta\} \tag{24}$$

Here d is the y coordinate of the sphere, and h the depth of the sphere. Notice that T^* is of the form

$$T_1^* = \sum_{n=0}^{\infty} B_n R^n P_n(\cos\theta) \tag{25}$$

$$B_0 = \frac{(T_2 h - T_1 z_2)}{(z_1 - z_2)},$$

$$B_1 = \frac{(T_1 - T_2)}{(h - z_2)}, \tag{26}$$

with $B_n = 0$, $n = 2, 3 \ldots\ldots$.

The set of equations for the unknown quantities A_n may be found by multiplying equation (20) by $\sin\theta P_n(\cos\theta)$ and integrating with respect to θ from 0 to π . The integrals can be evaluated by using the orthogonality properties of the Legendre functions. By this means one obtains equation (27).

$$b^m A_m = B_m + M\left\{\sum_{n=0}^{\infty}\frac{nb^n}{2n+1}\left\{A_n\delta_{mn} - \frac{2(-1)^{m+n}A_n}{n!m!}\left(-\frac{b}{2z_2}\right)^{m+n+1}(m+n)!\right\}\right\} \tag{27}$$

$$\delta_{mn} = 1, m = n$$
$$= 0, m \neq n$$

Equation (27) defines a system of simultaneous equations of infinite order for the A_n's, and so must be truncated. This is possible because only the lower order harmonics are important for the case where b/z_2 is small. By this means one finds estimates, $\bar{A}_n$'s, for the A_n's.

Once estimates, $\bar{A}_n$, for the coefficients A_n have been found from equation (23), by truncating the series after N terms, they are inserted into equation (19) to yield

$$T = T^* + \frac{m}{2\pi}\sum_{n=0}^{N}\frac{2\pi b^{n+1}n}{2n+1}\left\{\frac{\bar{A}_n P_n(\cos\theta)b^{n+1}}{R^{n+1}} - \frac{b}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{A}_n(-1)^n(\lambda b)^n}{n!}Q(z,x,y)\,d\alpha\, d\beta\right\} \tag{28}$$

The integral in equation (24) may be evaluated in closed form by changing the double integral from rectangular coordinates to cylindrical coordinates and then using a result of Weelan (1968 P.72 No. 1.408).

We find

$$T = T^* + \frac{M}{2\pi}\sum_{n=0}^{N}\frac{2\pi n b^{n+1}}{2n+1}\left\{\bar{A}_n P_n(\cos\theta)\frac{b^{n+1}}{R^{n+1}} - \frac{2A_n(-1)^n P_n(\cos\psi)}{1}\left(\frac{b}{Q}\right)^{n+1}\right\} \tag{29},$$

where

$$\cos\psi = \frac{(z-2z_2)}{\sqrt{x^2+y^2}}$$

$$Q = \sqrt{(z-2z_2)^2 + x^2 + y^2} \tag{30}$$

5. Discussion

For the commonly encountered problem of heat refraction about a heterogeneity in a layered ground, integral equation methods provide a convenient way of solving for the temperature distribution in the ground. Moreover analytic expressions can be found for solutions of this equation for simple shaped heterogeneities such as spherical ones.

The theory given here has its counterpart in the electrical geophysical prospecting and the modelling of magnetic anomalies where demagnetisation effects are important. For these cases the theory has been implemented in the form of computer programs, which are available. See Barnett (1972), Lee (1980). The shape of the heterogeneity is specified by means of a top, bottom, contour levels and a number of points on each contour. This information is sufficient to define the body in terms of a series of triangular facets. As the integral is taken over the surface rather than a volume, a larger body may be modelled, for a fixed number of unknowns, than is the case with other numerical methods.

The integral equation is solved for the body by assuming that the heat flux across each facet is constant. When one makes that assumption the surface integral of equation (8) is represented as the sum of a series of integrals over each facet. Next a set of simultaneous equations are found by applying the boundary condition to the heat flux at each facet. For further details, see Lee (1980) Barnett (1972).

For these cases the algorithm was checked against the analytical results for a spherical model and it was found that the numerical model gave accurate results.

As the Green's function for this case is of a similar form to that of an electrical heterogeneity in a layered ground, the existing software can be modified. This is possible because the Green's function for the electrical case is the sum of two terms rather than the difference of two as in the case with equation (19). That is to say the Green's function for the electrical case is the same as equation (19) and the minus sign has been altered to a positive one. As equation (18) shows the more general case is just the sum of higher order images. As these terms have the same structure as the second terms they present no numerical difficulty.

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